Journal of Algebra **221**, 50–59 (1999)
Article ID jabr.1999.7939, available online at http://www.idealibrary.com on IDEAL®

Interior Algebras and Varieties

A. V. Kelarev

School of Mathematics and Physics, University of Tasmania, G.P.O. Box 252-37, Hobart, Tasmania 7001, Australia
E-mail: kelarev@hilbert.maths.utas.edu.au

and

T. Stokes

School of Mathematical and Physical Sciences, Murdoch University, Perth, Western Australia 6150, Australia
E-mail: stokes@prodigal.murdoch.edu.au

Communicated by T. E. Hall

Received April 9, 1998

1. INTRODUCTION

We define interior algebras, which are natural generalizations of Boolean rings equipped with an interior operator and of equality algebras. We characterize interior algebras in terms of a distinguished subsemilattice. Fully interior algebras, in which the subsemilattice consists of all idempotents, are generalizations of Boolean rings and semilattices. Our main theorems describe all ring and semigroup varieties consisting of fully interior algebras.

2. INTERIOR ALGEBRAS

The notion of a Boolean ring with an interior operation has been around for some time: they are Boolean rings with identity on which an interior operation is defined, satisfying the usual Kuratowski sorts of conditions:

0021-8693/99 $30.00

$I(0) = 0$, $I(1) = 1$, $I(a) \leq a$, $I(I(a)) = I(a)$. and $I(a)I(b) = I(ab)$ for all $a, b \in R$. These were considered by McKinsey and Tarski in [8] (in terms of a closure operation, equivalent via $C(a) = 1 - I(a+1)$), where it was shown that every such interior ring can be embedded in the interior ring of all subsets of a topological space. Interior rings are also of significance in modal logic, where they provide the algebraic models for the so-called S_4 form of modal logic, probably the most important of the non-classical modal logics. Interior rings are fairly well understood; see [9], for instance.

The idea of an interior operation makes sense for semigroups and rings in general. We shall say that a semigroup or ring R is an *interior algebra* if it has a unary operation I which satisfies the following rules:

1. $I(a)I(b) = I(b)I(a)$,
2. $aI(a) = I(a)$,
3. $I(I(a)) = I(a)$, and
4. $I(a)I(b) = I(ab)I(b)$

for all $a, b \in R$.

Thus the class of interior semigroups is a variety of algebras if one views I as an operation; similarly for interior rings.

There is a term equivalence between the variety of interior rings with identity and rings which are equality structures, as is pointed out in [3]. Equality structures are an important special case of equality algebras, introduced in [2] in relation to artificial intelligence and definitional reasoning. The important special case of Boolean ring equality structures was in effect first considered by Suszko in [13], and is considered further in [4].

The definition of an interior algebra generalises the Boolean ring case: for the interior operation, the first rule is not needed if R is commutative, and the fourth is equivalent to $I(a)I(b) = I(ab)$ if R is a Boolean ring, as is easily checked. The reason for the form of the fourth rule will become clear shortly.

Recall that a *semilattice* is a commutative semigroup entirely consisting of idempotents. Every semilattice is a partially ordered set with respect to the natural order $e \leq f \Leftrightarrow ef = e$. If M is a subset of a semilattice S, then by $\max M$ we denote the largest element of M, if it exists.

In a Boolean interior ring R, the set $L = \mathrm{Im}(I)$ is a subsemilattice of the multiplicative semilattice of R by the final rule and is closed under I. For $a \in R$, $I(a) \leq a$, and if $I(b) \leq a$, then $I(b) = I(I(b)) \leq I(a)$ so $I(a) = \max\{\alpha \in L \mid \alpha \leq a\}$. Conversely, if a Boolean ring R has a multiplicative subsemilattice L which is such that $J(a) = \max\{\alpha \in L \mid \alpha \leq a\}$ exists for all $a \in R$, then one can verify that J is an interior operation on R. This all follows as a special case of the following theorem.

THEOREM 1. *Let R be a ring or a semigroup. Then I is an interior operation on R if and only if M has a multiplicative subsemilattice L such that $I(a) = \max\{\alpha \mid a\alpha = \alpha\}$ exists for all $a \in R$.*

Proof. Suppose I is an interior operation on R. Let $L = \mathrm{Im}(I)$. For all $\alpha, \beta \in L$, $\alpha\beta = I(\alpha)I(\beta) = I(\beta)I(\alpha) = \beta\alpha$, $\alpha\alpha = \alpha I(\alpha) = I(\alpha) = \alpha$ and $\alpha\beta = I(\alpha)I(\beta) = I(\alpha\beta)I(\beta) = I(\alpha\beta)\beta$, so multiplying both sides by α and using commutativity, $\alpha\beta = I(\alpha\beta)\alpha\beta = I(\alpha\beta)$. This means that L is a multiplicative subsemilattice of R. Now for all $a \in R$, $aI(a) = I(a)$, and if $aI(b) = I(b)$ for some $b \in R$, then $I(a)I(b) = I(a)I(I(b)) = I(aI(b))I(I(b)) = I(I(b))I(I(b)) = I(b)$, so $I(a) = \max\{\alpha \in L \mid a\alpha = \alpha\}$.

Conversely, suppose R has a multiplicative subsemilattice L such that $I(a) = \max\{\alpha \in L \mid a\alpha = \alpha\}$ exists for all $a \in R$. All the interior algebra rules are clear except the last two. But for all $a \in R$, $I(a)I(I(a)) = I(I(a))$ by definition, so $I(I(a)) \leq I(a)$, but also $I(a)I(a) = I(a)$ since elements of L are idempotent, so $I(a) \leq I(I(a))$ by definition. Thus $I(a) = I(I(a))$. Also, if $a\alpha = \alpha$ and $b\beta = \beta$ for some $a, b \in R$ and $\alpha, \beta \in L$, then

$$ab(\beta\alpha) = a(b\beta)\alpha = a\beta\alpha = (a\alpha)\beta = \alpha\beta,$$

so in particular, $I(a)I(b) \leq I(ab)$ so also $I(a)I(b) \leq I(ab)I(b)$; if also $ab\gamma = \gamma$ for some $\gamma \in L$, then $a\beta\gamma = a(b\beta)\gamma = (ab\gamma)\beta = \gamma\beta$, so in particular $I(ab)I(b) \leq I(a)I(b)$ and so $I(ab)I(b) \leq I(a)I(b)$. Thus $I(a)I(b) = I(ab)I(b)$. ∎

3. FULLY INTERIOR RINGS

In an interior algebra M, the largest L_M can be is all of $E(M)$, the set of all idempotents. Let us call a semigroup or ring in which this occurs a *fully interior algebra*. Thus R is a fully interior algebra if and only if $E(R)$ is a semilattice and, for each $r \in R$, there is a largest $e \in E(R)$ satisfying $re = e$. Clearly, all Boolean rings and all semilattices are fully interior algebras. It is of interest to describe fully interior algebras in the language of identities.

In this section we shall obtain a complete description of all ring varieties consisting of fully interior algebras. For the standard facts on rings satisfying polynomial identities we refer the reader to [10].

THEOREM 2. *For any ring variety V, the following conditions are equivalent*:

(i) *V consists of fully interior algebras*;

(ii) *V satisfies identities $x^n = x^{2n}$, $x^n y^n = y^n x^n$, $x^n(y - y^{n+1}) = 0$, and $(y - y^{n+1})x^n = 0$, for some $n > 0$.*

Proof. (ii) $\Rightarrow$ (i) Let V be a variety satisfying the identities $x^n = x^{2n}$, $x^n y^n = y^n x^n$, and $x^n(y - y^{n+1}) = (y - y^{n+1})x^n = 0$, where n is a positive integer.

Take any ring R in V. Since zero is the only idempotent of a quasiregular ring, the identity $x^n = x^{2n}$ implies that the Jacobson radical $\mathcal{J}(R)$ satisfies $x^n = 0$. The semisimple ring $R/\mathcal{J}(R)$ is a subdirect product of primitive rings which are in V. By the Jacobson density theorem every primitive ring P either is isomorphic to a matrix ring over a skew field, or for all $m \geq 1$ it contains subrings which can be homomorphically mapped onto $m \times m$ matrix rings over skew fields. If $m > 1$, then every $m \times m$ matrix ring over a skew field fails to satisfy $x^n y^n = y^n x^n$, because it contains noncommuting idempotents. Therefore all primitive rings in V are skew fields. However, in view of the identity $x^n = x^{2n}$ every skew field in V satisfies $x = x^{n+1}$, and so it is a finite field. Hence $R/\mathcal{J}(R)$ is a subdirect product of finite fields.

Therefore $R/\mathcal{J}(R)$ does not contain nonzero nilpotent elements. Thus $\mathcal{J}(R)$ is equal to the set of all nilpotent elements of R.

Denote by I the ideal generated by $E(R)$ in R.

For any $r \in R$, $r = r^{n+1} + (r - r^{n+1})$. Clearly, $r - r^{n+1}$ is a nilpotent (because $(r - r^{n+1})r^{n-1} = 0$), and so it is in $\mathcal{J}(R)$. Also, r^{n+1} belongs to I, because r^n is an idempotent. Thus $R = I + \mathcal{J}(R)$.

Take any $x \in I$, say $x = \sum_{i=1}^n a_i e_i b_i$, where $a_i, b_i \in R^1$ and $e_i \in E(R)$. For any $1 \leq i \leq n$ and $y \in \mathcal{J}(R)$, we get $b_i y \in \mathcal{J}(R)$, so $(b_i y)^n = 0$. Hence $a_i e_i (b_i y) = a_i e_i^n (b_i y - (b_i y)^{n+1}) = 0$ in view of the third identity of V. Therefore $xy = 0$. Similarly, $yx = 0$ by the fourth identity of V. Thus $I\mathcal{J}(R) = \mathcal{J}(R)I = 0$.

For any $x, y \in R$, we get $(xy^n)^n = (xy^n)^{2n}$, and so $d = xy^n - (xy^n)^{n+1}$ is a nilpotent. Hence $d = dy^n \in \mathcal{J}(R)I = 0$, and so $xy^n - (xy^n)^{n+1} = 0$.

Take $x \in I \cap \mathcal{J}(R)$. Then $x = \sum_{i=1}^n a_i e_i b_i$ for some $e_i \in E(R)$, $a_i, b_i \in R^1$. The identity $xy^n - (xy^n)^{n+1} = 0$ shows that $a_i e_i = (a_i e_i)^{n+1} = f_i a_i e_i$, where $f_i = (a_i e_i)^n$ is an idempotent. Take an idempotent

$$f = \sum_{i=1}^n f_i - \sum_{1 \leq i < j \leq n} f_i f_j + \sum_{1 \leq i < j < k \leq n} f_i f_j f_k + \cdots + (-1)^{n-1} f_1 f_2 \cdots f_n.$$

Since all idempotents in R^1 commute by the second identity of V, we get

$$f = f_i + \left[\sum_{j \neq i} f_j - \sum_{j<k;\, j,\, k \neq i} f_j f_k + \cdots + (-1)^{n-1} f_1 \cdots f_{i-1} f_{i+1} \cdots f_n \right] (1 - f_i).$$

Therefore $ff_i = f_i$ for all i. Hence $x = fx \in I\mathcal{J}(R) = 0$. This means that R is a direct sum of I and $\mathcal{J}(R)$.

A direct sum of two rings is a fully interior algebra if and only if both its components are fully interior algebras. Obviously, the nil ring $\mathcal{J}(R)$ is a fully interior algebra. It remains to show that the same can be said of $P = R/\mathcal{J}(R)$.

We have already proved that P is a subdirect product of fields which are in V. Suppose that P is contained in the direct product $D = \prod_{i\in I} F_i$ of fields $F_i \in V$.

Take any element $r \in P$. Put $L = \{f \in E(D) \mid rf = f\}$ and choose any $e \in L$. Denote by r_i and e_i the projections of r and e on F_i. Let 1_i be the identity of F_i. If $r_i = 1_i$, then $e_i \in \{1_i, 0\}$. If $r_i \neq 1_i$, then $e_i = 0$. Since $1_i > 0$, it follows that the largest idempotent f in L has projections

$$f_i = \begin{cases} 1_i & \text{if} r_i = 1_i \\ 0 & \text{otherwise.} \end{cases}$$

If $r_i = 0$ then trivially $r_i - r_i(r_i - r_i^2)^n = 0 = f_i$. Further, if $r_i = 1_i$, then $r_i - r_i(r_i - r_i^2)^n = 1_i$. If $r_i \notin \{0, 1_i\}$, then $r_i \neq r_i^2$, and so $(r_i - r_i^2)^n = 1_i$ in view of the identity $x = x^{n+1}$ in the field F_i; whence $r_i - r_i(r_i - r_i^2)^n = 0 = f_i$, again. Hence $f = r - r(r - r^2)^n$. Therefore $f \in P$. It follows that $f = \max\{e \in E(P) \mid re = e\}$. Thus P is a fully interior algebra, and so (i) holds.

(i) $\Rightarrow$ (ii) Let V be a variety consisting of fully interior rings. A ring variety is called *periodic* if it satisfies an identity $x^a = x^{a+b}$, for some positive integers a, b. All periodic varieties are described in [14]. In particular, it is proved that a variety is periodic if and only if it does not contain the variety $V_p = [px = xy - yx = 0]$ for each prime p.

Consider the commutative ring $K = \mathbb{Z}_p[x, y]/(y - y^2, y - xy)$, its direct power K^∞, and the subring S generated in the direct power by $a = (x, x, x, \ldots)$, and all $y_1 = (y, 0, 0, \ldots)$, $y_2 = (0, y, 0, \ldots)$, $y_3 = (0, 0, y, \ldots)$, ... Then the set $\{e \in E(S) \mid ae = e\}$ contains all idempotents $\sum_{i=1}^n y_i$, and so it does not have a largest idempotent. Thus S is not a fully interior algebra. Clearly, the ring S belongs to V_p. Therefore our variety V does not contain V_p, and so it satisfies the identity $x^a = x^{a+b}$ for some positive integers a, b. Hence x^{ab} is an idempotent for every x. Thus V satisfies the identity $x^n = x^{2n}$, where $n = ab$.

Given that the idempotents of every ring in V form a semilattice, it follows that V satisfies the identity $x^n y^n = y^n x^n$.

Take any ring R in V. Since zero is the only idempotent of a quasiregular ring, we see that the Jacobson radical $\mathcal{J}(R)$ satisfies the identity $x^n = 0$. The semisimple ring $R/\mathcal{J}(R)$ is a subdirect product of primitive rings which are in V. If $m > 1$, then every $m \times m$ matrix ring over a skew field contains noncommuting idempotents, and so it does not belong to V. It follows from the Jacobson density theorem that every primitive ring P in the subdirect product representation of $R/\mathcal{J}(R)$ is a skew field. The identity $x^n = x^{2n}$ shows that P satisfies $x = x^{n+1}$. Therefore P is a finite field. Thus $R/\mathcal{J}(R)$ is a subdirect product of finite fields. In particular, it has no nonzero nilpotent elements. It follows that $\mathcal{J}(R)$ is equal to the set of all nilpotent elements of R.

Look at the free ring F_2 of rank 2 in V. Let x, y be the free generators of F_2. If $x^n = 0$, then V satisfies the identity $x^n = 0$, and all the required identities in (ii) follow immediately. Suppose that $x^n \neq 0$. If $y - y^{n+1} = 0$, then V satisfies the identity $y - y^{n+1} = 0$, and so (ii) holds again. So suppose that $y - y^{n+1} \neq 0$.

Further, suppose to the contrary that $x^n(y - y^{n+1}) \neq 0$.

First, assume that $x^n(y - y^{n+1})x^n = 0$. Consider the element $c = x^n + x^n(y - y^{n+1})$. We get $c^2 = c$, $x^n c = c$, and $cx^n = x^n$. Therefore the distinct idempotents c and x^n do not commute. It follows that $x^n(y - y^{n+1})x^n \neq 0$.

As we have proved, the set of nilpotent elements of every ring in V is an ideal. Since $y - y^{n+1}$ is a nilpotent because of the identity $x^n = x^{2n}$, we see that $x^n(y - y^{n+1})x^n$ is a nilpotent, too. Let k be a positive integer such that $(x^n(y - y^{n+1})x^n)^k \neq 0$ and $(x^n(y - y^{n+1})x^n)^{k+1} = 0$. Put $c = x^n$, $d = (x^n(y - y^{n+1})x^n)^k \neq 0$. Then $d^2 = 0$, $c^2 = c$, and $cd = dc = d$.

Consider the subring A generated by c and d in F_2. Clearly, the subring $\langle d \rangle$ generated by d is an ideal in $A \in V$.

If $md \neq 0$ for all positive integers m, then we can factor out the ideal generated by $2c, 2d$ from A and assume that $2c = 2d = 0$. If $md = 0$ for some m, then we may assume that m is a minimal positive integer with this property, and we can take a prime divisor p of m, factor out the ideal generated by $\frac{m}{p}c$, $\frac{m}{p}d$, and assume that $pc = pd = 0$. In any case we may assume that $pc = pd = 0$ for a prime p.

For $i = 1, 2, \ldots$, denote by A_i an isomorphic copy of A with generators c_i, d_i corresponding to c, d. Then the direct product $D = \prod_{i=1}^{\infty} A_i$ belongs to V. If $r \in D$, $i \in \mathbb{N}$, then we denote by r_i the projection of r on A_i, and we write $r = (r_1, r_2, \ldots)$.

Consider the subring Q of D generated by the union of all subrings A_i, the direct product $\prod_{i=1}^{\infty} \langle d_i \rangle$, and the element $(c_1, c_2, \ldots)$. Look at the element

$$a = (c_1, c_2 + d_2, c_3, c_4 + d_4, c_5, \ldots)$$

of Q. Suppose that $f = (f_1, f_2, \ldots)$ is an idempotent of D such that $af = f$. Take any $k > 0$.

Clearly, c_{2k-1} is the largest idempotent of A_{2k-1} such that $a_{2k-1}c_{2k-1} = c_{2k-1}$. Thus $f_{2k-1} \leq c_{2k-1}$.

Further, assume that $f_{2k} = mc_{2k} + ld_{2k}$. Since $f_{2k} = f_{2k}^2$, we get $f_{2k}^2 = (m^2)c_{2k} + (2ml)d_{2k}$. Hence $mc_k = m^2 c_k$, and so p divides m or $m - 1$. If p divides $m - 1$, then $a_{2k} f_{2k} = f_{2k}$ implies $c_{2k} + (m + l)d_{2k} = c_{2k} + ld_{2k}$ and

p divides $m = m + l - l$, so $p|1$, a contradiction. Therefore p divides m, whence $f_{2k} = ld_{2k}$ is a nilpotent, and so $f_{2k} = 0$.

It follows that

$$g = (c_1, 0, c_3, 0, c_5, \ldots)$$

is the largest idempotent of D satisfying $ag = g$. However, g is not in Q. Denote by L the set of all idempotents r of Q such that $ar = r$. Then it is easily seen that L contains all idempotents $f = (f_1, f_2 \ldots)$ such that, for $k = 1, 2, \ldots$, $f_{2k-1} \in \{0, c_{2k-1}\}$ and $f_{2k} = 0$, where only a finite number of the elements f_{2k-1} are nonzero by the definition of Q. The least idempotent of D which is greater than all these idempotents is g, which is not in Q. Therefore L does not have a largest element.

This contradiction shows that V satisfies $x^n(y - y^{n+1}) = 0$. Similarly, it satisfies $(y - y^{n+1})x^n = 0$. This completes our proof. ∎

In particular, it follows that all semisimple varieties consist of fully interior rings. Semisimple varieties form an important class considered by many authors, see [7] for references. Identities of semisimple varieties were characterized in [5].

Our proof shows that all rings in varieties described in Theorem 2 are direct products of their radicals and some semisimple rings. All varieties with the property that the Jacobson radical of every finitely generated member is a semidirect summand were characterized in [15].

4. FULLY INTERIOR SEMIGROUPS

Our next theorem describes all semigroup varieties consisting of fully interior algebras. For preliminaries on semigroups and semigroup varieties we refer to [6] and [11, 12], respectively.

THEOREM 3. *For any semigroup variety V, the following conditions are equivalent*:

(i) *V consists of fully interior algebras*;

(ii) *V satisfies the identities $x^n = x^{n+1}$ and $x^n y^n = y^n x^n$, for some* $n > 0$.

First, we prove the following lemma.

LEMMA 4. *For any semigroup S, the following conditions are equivalent*:

(i) *all subsemigroups of S are fully interior algebras*;

(ii) *for every* $x \in S$, *there exists a positive integer* $n(x)$ *such that* $x^{n(x)} = x^{n(x)+1}$ *and and* $x^{n(x)}y^{n(x)} = y^{n(x)}x^{n(x)}$ *for all* $x, y \in S$.

Proof. (ii) $\Rightarrow$ (i) Suppose that $x^{n(x)} = x^{n(x)+1}$ and $x^{n(x)}y^{n(x)} = y^{n(x)}x^{n(x)}$, for all $x, y \in S$. Take any subsemigroup T of S.

If $e, f \in E(T)$, then we get $ef = e^{n(e)}f^{n(f)} = f^{n(f)}e^{n(e)} = fe$, and so $E(T)$ is a semilattice.

Pick any element a in T. Let $e = a^{n(a)}$. Clearly, $ae = a^{n(a)+1} = e = ea$. On the other hand, look at any $f \in E(T)$ such that $af = f$. We get $f = a^{n(a)}f = ef$, and so $f \leq e$. Therefore e is the largest element in the set $\{e \in E(T) \mid ae = e\}$. Thus T is a fully interior algebra.

(i) $\Rightarrow$ (ii) Obviously, nontrivial groups are not fully interior algebras. Therefore (i) implies that S contains only groups of order 1. Infinite cyclic semigroups are not fully interior algebras, and so S is periodic. It follows that every monogenic semigroup in S is nil, i.e., for every $x \in S$ there exists $n(x)$ such that $x^{n(x)} = x^{n(x)+1}$, where $n(x) > 0$.

If $x, y \in S$, then $x^{n(x)}$, $y^{n(x)}$ are idempotents, and we get $x^{n(x)}y^{n(x)} = y^{n(x)}x^{n(x)}$, as required. ∎

Proof of Theorem 3. (ii) $\Rightarrow$ (i) If V satisfies the identities $x^n = x^{n+1}$ and $x^n y^n = y^n x^n$, for some $n > 0$, then Lemma 4 shows that all semigroups in V are fully interior algebras.

(i) $\Rightarrow$ (ii) Consider the free semigroup F_2 of rank 2 in V with free generators x, y. By Lemma 3 there exist positive integers $n(x), n(y)$ such that $x^{n(x)} = x^{n(x)+1}$ and $x^{n(x)}y^{n(y)} = y^{n(y)}x^{n(x)}$. Put $n = n(x)n(y)$. Then is follows that V satisfies the identities $x^n = x^{n+1}$ and $x^n y^n = y^n x^n$. ∎

A semigroup S is said to be *Archimedean* if, for any two elements of S, each divides some power of the other. Every commutative semigroup S has a unique representation as a semilattice of its Archimedean components (see [1, Sect. 4.3]). This means that there exist a semilattice Y and Archimedean subsemigroups S_y, $y \in Y$, such that S is a disjoint union of the S_y, and $S_x S_y \subseteq S_{xy}$ for all $x, y \in Y$.

We shall consider commutative semigroups with all Archimedean components containing idempotents. This class is quite large. In particular, it includes all periodic and all regular commutative semigroups.

THEOREM 5. *Let* S *be a commutative semigroup all of whose Archimedean components contain idempotents. Then the following conditions are equivalent*:

(i) S *is a fully interior algebra*;

(ii) *for every cyclic subgroup* G *of* S *there exists a largest idempotent* $e \in S$ *such that* $eG = \{e\}$.

Proof. Let $S = \bigcup_{y \in Y} S_y$ be a semilattice of Archimedean components and let e_y be an idempotent of S_y. Every S_y has a unique idempotent by [1, Sect. 4.3, Exercise 2], , and the product $G_y = e_y S_y$ is a subgroup ideal of S_y such that S_y/G_y is nil [1, Sect. 4.3, Exercise 3].

(i) $\Rightarrow$ (ii) Let S be a fully interior algebra. Take any nontrivial cyclic subgroup G of S. Suppose that G is generated by a. Clearly, G is contained in one Archimedean component S_y of S. Then e_y is the identity of G.

Compare the sets $A = \{e \in E(S) \mid ae = e\}$ and $B = \{e \in E(S) \mid eG = \{e\}\}$.

If $e \in A$, then $a^m e = e$ for all m. Hence $|eG| = 1$. Further, if $e \in B$, then $eG = \{e\}$ implies $ae = e$. Thus $A = B$. It follows that B has a largest element, as required.

(ii) $\Rightarrow$ (i) Suppose that for every cyclic subgroup G of S there exists a largest idempotent $e \in S$ such that $eG = \{e\}$.

Take any element $a \in S$. Suppose that $a \in S_x$. Put $f = e_x$. The product af belongs to the group $G_x = fS \cap S_x$. Denote by G the group generated by af. Clearly, f is the identity of G.

Now compare the sets $A = \{e \in E(S) \mid ae = e\}$ and $B = \{e \in E(S) \mid eG = \{e\}\}$.

Let $e \in A$. There exists $n > 0$ such that $a^n \in fS$, and so $a^n f = a^n$. Therefore $ae = e$ implies $ef = eaf = ea^n f = ea^n = e$, and so $e \leq f$. Hence $(af)e = e$ and we get $(af)^m e = e$ for all m. Therefore $eG = \{e\}$, and so $e \in B$.

If $e \in B$, then $(af)e = fe = e$, and we get $ae = e$. Therefore $e \in A$.

Thus $A = B$. It follows that A contains a largest element, which completes the proof. ∎

Next we give an example of a finite commutative inverse semigroup which is not a fully interior algebra.

EXAMPLE 6. Let $S = \{0, a, b, c, 1\}$ be a commutative monoid with zero 0, identity 1 and multiplication defined by $ab = 0$, $a^2 = ac = a$, $b^2 = bc = b$, and $c^2 = 1$. Clearly, S is an inverse semigroup with Archimedean components $\{0\}$, $\{a\}$, $\{b\}$, and $\{c, 1\}$. The set $\{e \in E(S) \mid ce = e\}$ is equal to $\{0, a, b\}$. It does not have a largest idempotent. Thus S is not a fully interior algebra.

This example shows that even the class of commutative semigroups which are fully interior algebras is quite complicated. It is clear that a completely 0-simple semigroup is a fully interior algebra if and only if it is inverse (see [6, Theorems 3.3.3 and 5.1.8]). Straightforward verification shows that the bicyclic semigroup is a fully interior algebra, too (see [6, Sect. 1.6].

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